Last time we discussed linear functions

(a.k.a. maps / transformations)

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ means $f(v+v') = f(v) + f(v')$
and $f(cv) = c f(v)$

(examples: rotations, reflections, rescalings, projections, shearings)

THM 5.5: any linear function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

can be written uniquely as

$$f(x) = Ax \quad , \forall x \in \mathbb{R}^n$$

for some $m \times n$ matrix A depending on f

Conversely: all functions of this sort are linear
(the converse was proved last time, at the end of class)

Proof: "standard basis vectors" of $\mathbb{R}^n$

$$e_1=\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}\quad e_2=\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}\quad\cdots\cdots\quad e_n=\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}$$

$\forall\ x=\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}\in\mathbb{R}^n$ can be written very easily as

$$x=\begin{pmatrix}x_1\\0\\\vdots\\0\end{pmatrix}+\begin{pmatrix}0\\x_2\\0\\\vdots\\0\end{pmatrix}+\cdots+\begin{pmatrix}0\\\vdots\\0\\x_n\end{pmatrix}=x_1\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}+x_2\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}+\cdots+x_n\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$$

$$\boxed{x=x_1e_1+\cdots+x_ne_n}\quad,\ x_1,\dots,x_n\in\mathbb{R},\ e_1,\dots,e_n\in\mathbb{R}^n$$

Recall that $f:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ is a linear functions

$$\underset{\mathbb{R}^n}{f(x)}=f(x_1e_1+\cdots+x_ne_n)\overset{f\text{ linear}}{=\!=\!=}x_1f(e_1)+\cdots+x_nf(e_n)$$

linear combination of

$A_1=f(e_1)\ ,\dots,A_n=f(e_n)\in\mathbb{R}^m$

$$f(x)=x_1A_1+\cdots+x_nA_n$$

$$f(x)=Ax$$

$\square$

$A=(A_1\ \dots\ A_n)\in\mathbb{R}^{m\times n}$

$x=\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$

Constructive proof: A is actually being constructed, also unique

<u>Ex:</u> $f:\mathbb{R}^3\longrightarrow\mathbb{R}^2\ ,\ f\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}7x_1-5x_2\\9x_2+13x_3\end{pmatrix}$

how to write $f(x)=Ax$? Who should A be?

$\mathbb{R}^{2\times 3}$

$$A_1 = f(e_1) = f\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}7\\0\end{pmatrix}$$

$$A_2 = f(e_2) = f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-5\\9\end{pmatrix}$$

$$A_3 = f(e_3) = f\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\13\end{pmatrix}$$

$$\Longrightarrow A = \left[A_1\ A_2\ A_3\right] = \begin{pmatrix}7 & -5 & 0\\0 & 9 & 13\end{pmatrix}$$

Check that this works: $Ax = \begin{pmatrix}7 & -5 & 0\\0 & 9 & 13\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}7x_1 - 5x_2 + 0x_3\\0x_1 + 9x_2 + 13x_3\end{pmatrix}$

$\Downarrow$

$f(x) = Ax$ by comparison under the green arrow above

General strategy for determining if a $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is

- linear: check axioms $f(cv + c'v') = cf(v) + c'f(v')$, or $\forall v, v', x \in \mathbb{R}^n$, $\forall c, c' \in \mathbb{R}$ $\Updownarrow$ find $A \in \mathbb{R}^{m\times n}$ s.t. $f(x) = Ax$

- not linear: find some vectors v and v' s.t. $f(v+v') \neq f(v) + f(v')$ or find some $v \in \mathbb{R}^n$ and some $c \in \mathbb{R}$ s.t. $f(cv) \neq cf(v)$ (or show that $f(0) \neq 0$)

$$f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x^2\\y\end{pmatrix}$$

$$f\begin{pmatrix}-x\\-y\end{pmatrix} = \begin{pmatrix}(-x)^2\\-y\end{pmatrix}$$

these vectors are not opposites for general $x, y \in \mathbb{R}$ so f violates this property $\rightsquigarrow$

any linear function has the property that $f(-v) + f(v) = f(-v+v) = f(0) = 0$

$\Updownarrow$

$f(-v) = -f(v)$

What to do with linear functions?

$\forall f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$f, g : \mathbb{R}^n \to \mathbb{R}^m$

we can add them

$$f+g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$(f+g)(v) = f(v) + g(v) \quad , \forall v \in \mathbb{R}^n$$

we can also scalar multiply (given $c \in \mathbb{R}$)

$$cf : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$(cf)(v) = c f(v) \quad , \forall v \in \mathbb{R}^n$$

<u>Proposition</u>: if f and g are linear, then $f+g$ and cf are also linear.

<u>Proof</u>: (for $f+g$, do cf yourselves); we must check that far left & right are equal

- $(f+g)(v+v') = f(v+v') + g(v+v') \overset{f,g \text{ linear}}{=\!=} f(v) + f(v') + g(v) + g(v') = (f+g)(v) + (f+g)(v')$
- $(f+g)(cv) = f(cv) + g(cv) \overset{f \text{ linear}}{=\!=} c f(v) + c g(v) = c (f+g)(v)$

if $f(v) = Av$ for $A, B \in \mathbb{R}^{m \times n}$

$g(v) = Bv$

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then $(f+g)(v) = (A+B)v$

$(cf)(v) = (cA)v$

$f \rightsquigarrow A$

$g \rightsquigarrow B$

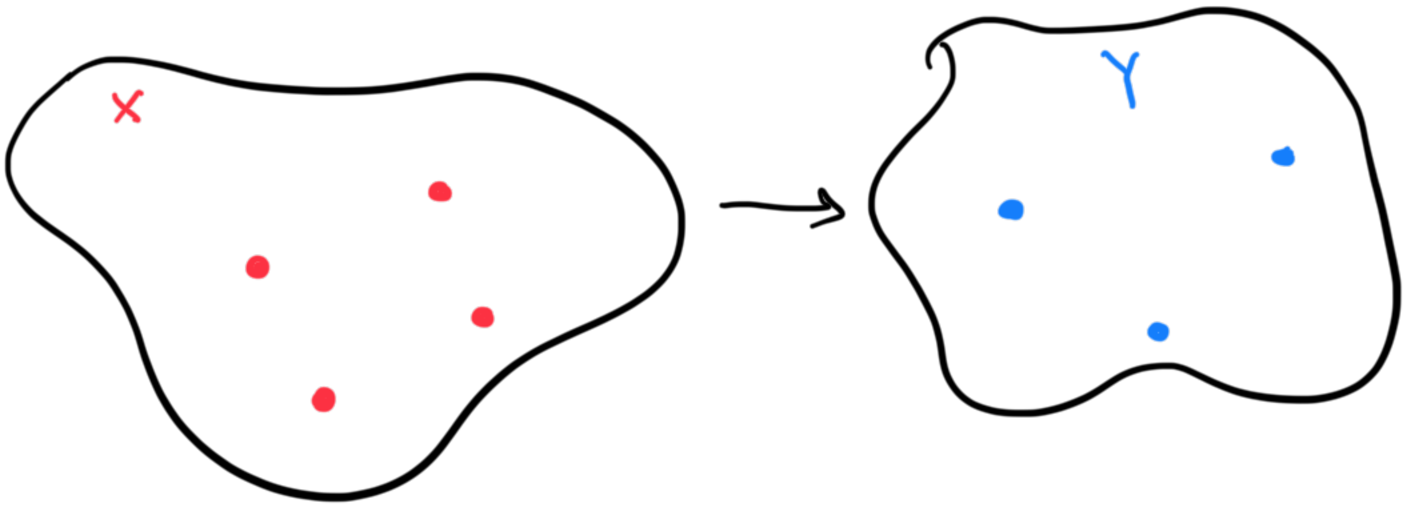
$\overset{\text{according to}}{\underset{\text{previous Thm}}{\Longrightarrow}}$

$f+g \rightsquigarrow A+B$

$cf \rightsquigarrow cA$

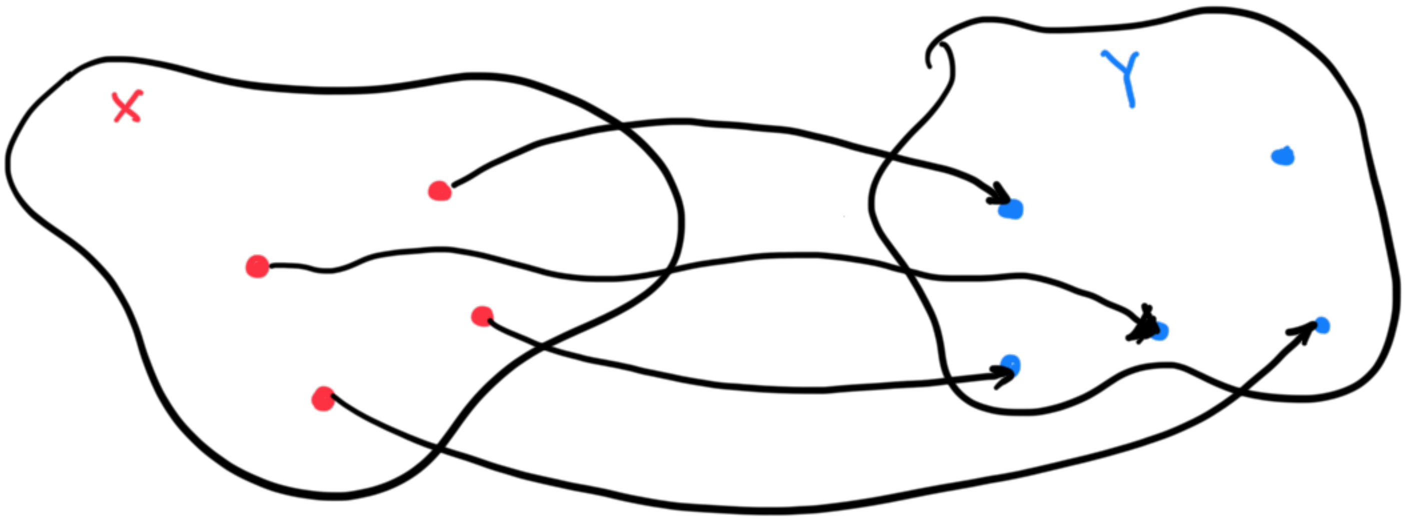
Injective, Surjective, Bijective function

$f:$



DEF 6.1: f is injective if it sends different elements of X to different elements of Y

$f:$



$\forall\ x \neq x'$ in X, we have $f(x) \neq f(x')$

$\Updownarrow$

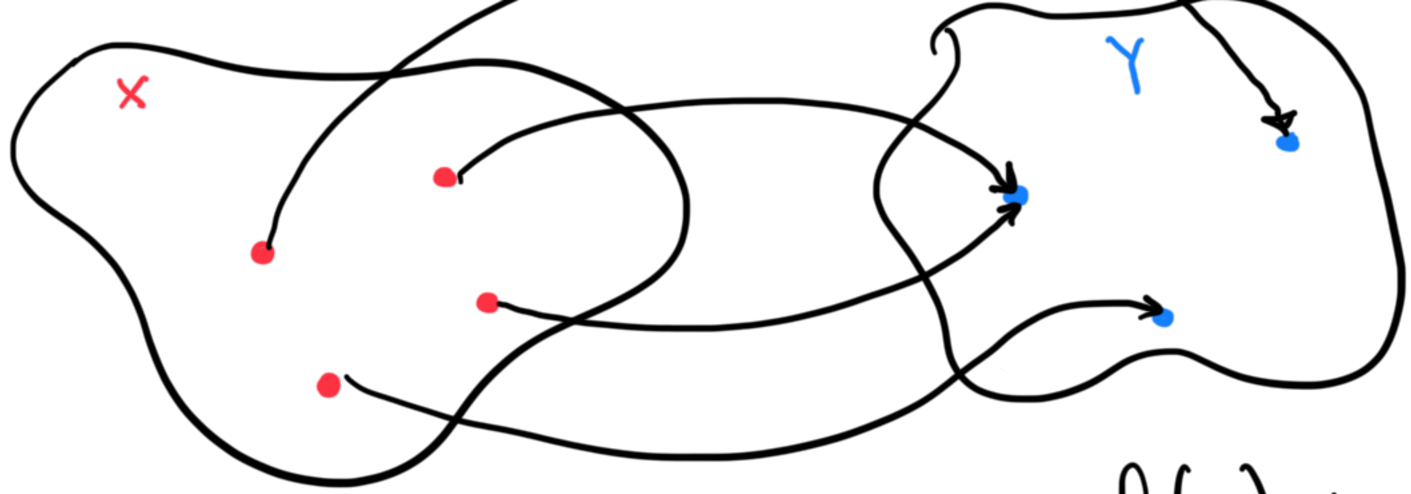
if $f(x) = f(x')$, then $x = x'$

To have a chance at f being injective, must have

size $X \leqslant$ size Y

DEF 6.2: f is called surjective if every element of Y is mapped onto by at least one element of X

$f:$



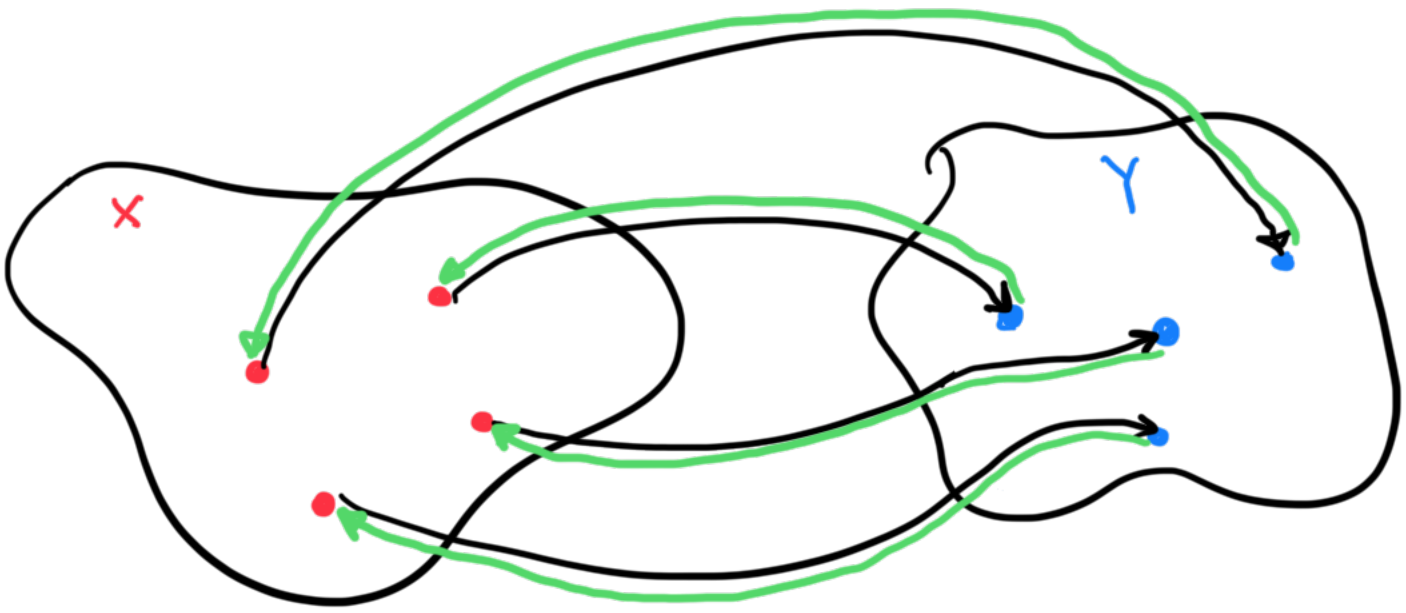
$\forall y \in Y, \exists x \in X$ s.t. $f(x) = y$

To have a chance at f being surjective, must have

size $X \geq$ size Y

DEF 6.3: f is called bijective if both $\begin{cases} \text{injective} \\ \text{surjective} \end{cases}$

$f:$



$: f^{-1}$

(f is bijective $\iff$ f is invertible)

To have a chance at f being bijective, must have

size $X =$ size Y

Which linear functions $f: \mathbb{R}^n \to \mathbb{R}^m$ are $\begin{cases} \text{inj} \\ \text{surj} \\ \text{bij} \end{cases}$?

THM 6.4 : assume $f(x) = Ax$ for some $A \in \mathbb{R}^{m\times n}$

(1) f is injective $\iff$ columns of A are linearly independent
$\iff$ REF(A) has n pivots ($m \geq n$)

(2) f is surjective $\iff$ columns of A span $\mathbb{R}^m$
$\iff$ REF(A) has m pivots ($m \leq n$)

(3) f is bijective $\iff$ columns of A are linearly independent and span $\mathbb{R}^m$
$\iff$ $m = n$ and REF(A) has $m = n$ pivots

(for a general $m \times n$ matrix, # pivots $\leq m, n$)

$$\begin{array}{cc} & n \\ m & \begin{pmatrix} \boxed{1} & * & 0 & 0 & * \\ 0 & 0 & \boxed{1} & 0 & * \\ 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

<u>Proof</u>: (3) is immediate from (1) & (2)

(2) is almost obvious, because

f is surjective $\iff$ $\forall\, b \in \mathbb{R}^m \;\exists\, x \in \mathbb{R}^n$ s.t. $f(x) = b$

$\iff$ $\forall\, b \in \mathbb{R}^m \;\exists\, x \in \mathbb{R}^n$ s.t. $Ax = b$

$\overset{\text{already proved}}{\iff}$ columns of A span $\mathbb{R}^m$

(1) f is injective $\iff$ $f(v) = f(v')$ implies $v = v'$

$\iff$ $Av = Av'$ implies $v = v'$

$\iff Av = Av'$ implies ...

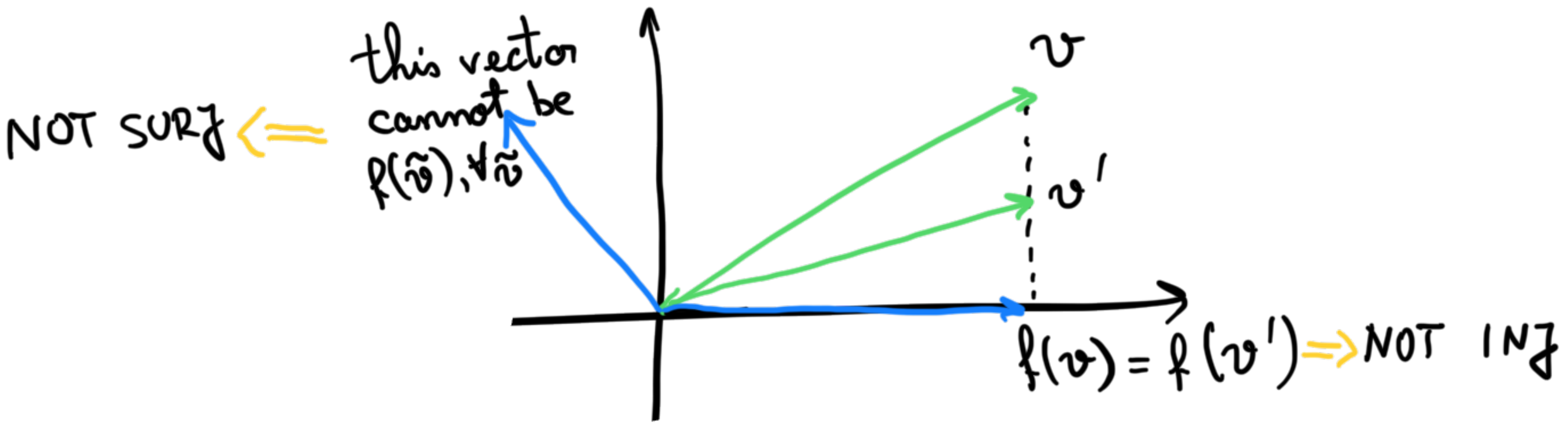
$\iff A(v-v') = 0$ implies $v - v' = 0$ denote $x = v - v'$

$\iff Ax = 0$ implies $x = 0$

$\iff$ homogeneous equation $Ax = 0$ has 1 solution, i.e. $x = 0$

a few days ago
$\iff$ columns of A are linearly independent

Ex: projection onto x-axis as a function $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$



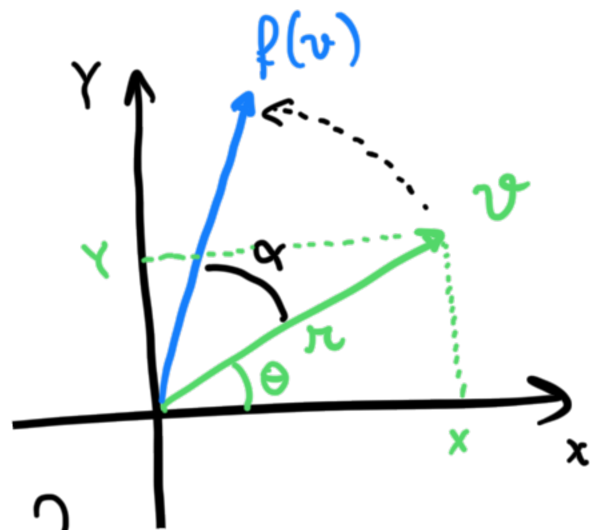
$$f: \mathbb{R}^2 \to \mathbb{R}^2, \quad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} \boxed{1} & 0 \\ 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$m = 2 > \#$ pivots
$n = 2 > \#$ pivots

but $$f: \mathbb{R}^2 \to \mathbb{R}^1, \quad f\begin{pmatrix} x \\ y \end{pmatrix} = (x) = \underbrace{\begin{pmatrix} \boxed{1} & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$m = 1 = \#$ pivots
$n = 2 > \#$ pivots

Rotation by a given angle α
(bijective because invertible)



$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$f\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$ how to determine A?

$f\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$; how to determine A?

recall polar coordinates $\begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow (r, \theta)$ means that

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \implies \begin{pmatrix} x \\ y \end{pmatrix} = r\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

geometrically obviously, $f\begin{pmatrix} x \\ y \end{pmatrix} = r\begin{pmatrix} \cos(\theta+\alpha) \\ \sin(\theta+\alpha) \end{pmatrix}$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = r\begin{pmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha \\ \cos\theta\sin\alpha + \sin\theta\cos\alpha \end{pmatrix}$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ \underbrace{r\cos\theta}_{x}\sin\alpha + \underbrace{r\sin\theta}_{y}\cos\alpha \end{pmatrix}$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\alpha - y\sin\alpha \\ x\sin\alpha + y\cos\alpha \end{pmatrix}$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Upshot: $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ is the matrix of rotation by α